

Labeling outerplanar graphs with maximum degree three

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Abstract

An $L(2, 1)$ -labeling of a graph G is an assignment of a nonnegative integer to each vertex of G such that adjacent vertices receive integers that differ by at least two and vertices at distance two receive distinct integers. The span of such a labeling is the difference between the largest and smallest integers used. The λ -number of G , denoted by $\lambda(G)$, is the minimum span over all $L(2, 1)$ -labelings of G . Bodlaender *et al.* conjectured that if G is an outerplanar graph of maximum degree Δ , then $\lambda(G) \leq \Delta + 2$. Calamoneri and Petreschi proved that this conjecture is true when $\Delta \geq 8$ but false when $\Delta = 3$. Meanwhile, they proved that $\lambda(G) \leq \Delta + 5$ for any outerplanar graph G with $\Delta = 3$ and asked whether or not this bound is sharp. In this paper we answer this question by proving that $\lambda(G) \leq \Delta + 3$ for every outerplanar graph with maximum degree $\Delta = 3$. We also show that this bound $\Delta + 3$ can be achieved by infinitely many outerplanar graphs with $\Delta = 3$.

Key words: $L(2, 1)$ -labeling; outerplanar graphs; λ -number

AMS subject classification: 05C15, 05C78

1 Introduction

In the channel assignment problem [11] one wishes to assign channels to transmitters in a radio communication network such that the bandwidth used is minimized whilst interference is avoided as much as possible. Various constraints have been suggested to put on channel separations between pairs of transmitters within certain distances, leading to several important optimal labeling problems which are generalizations of the ordinary graph coloring problem. Among them the $L(2, 1)$ -labeling problem [10] has received most attention in the past two decades.

Given integers $p \geq q \geq 1$, an $L(p, q)$ -labeling of a graph $G = (V(G), E(G))$ is a mapping f from $V(G)$ to the set of nonnegative integers such that $|f(u) - f(v)| \geq p$ if u and v are adjacent in G , and $|f(u) - f(v)| \geq q$ if u and v are distance two apart in G . The integers used by f are called the *labels*, and the *span* of f is the difference between the largest and smallest labels used by f . The $\lambda_{p,q}$ -number of G , $\lambda_{p,q}(G)$, is the minimum span over all $L(p, q)$ -labelings of G . We may assume without loss of generality that the smallest label used is 0, so that $\lambda_{p,q}(G)$ is equal to the minimum value among the largest labels

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used by $L(p, q)$ -labelings of G . In particular, $l(G) = l_{2,1}(G)$ is called the l -number of G . For a nonnegative integer k , a k - $L(2, 1)$ -labeling is an $L(2, 1)$ -labeling with maximum label at most k . Thus $\lambda(G)$ is the minimum k such that G admits a k - $L(2, 1)$ -labeling.

The $L(p, q)$ -labelling problem has received extensive attention over many years especially in the case when $(p, q) = (2, 1)$ (see [5] for a survey). Griggs and Yeh [10] conjectured that $\lambda(G) \leq \Delta^2$ for any graph G with maximum degree $\Delta \geq 2$. This has been confirmed for several classes of graphs, including chordal graphs [16], generalized Petersen graphs [8], Hamiltonian graphs with $\Delta \leq 3$ [13], etc. Improving earlier results, Goncalves [9] proved that $\lambda(G) \leq \Delta^2 + \Delta - 2$ for any graph G with $\Delta \geq 2$. Recently, Havet, Reed and Sereni [12] proved that for any $p \geq 1$ there exists a constant $\Delta(p)$ such that every graph with maximum degree $\Delta \geq \Delta(p)$ has an $L(p, 1)$ -labelling with span at most Δ^2 . In particular, this implies that the Griggs-Yeh conjecture is true for any graph with sufficiently large Δ .

The λ -number of a graph relies not only on its maximum degree but also on its structure. It is thus important to understand this invariant for different graph classes. In particular, for the class of planar graphs, Molloy and Salavatipour [15] proved that $l_{p,q}(G) \leq q\lceil 5\Delta/3 \rceil + 18p + 77q - 18$, which yields $l(G) \leq \lceil 5\Delta/3 \rceil + 77$, and Bella *et al.* [1] proved that the Griggs-Yeh conjecture is true for planar graphs with $\Delta \neq 3$. (See [1] for a brief survey on the $L(p, q)$ -labeling problem for planar graphs.) This indicates that the class of planar graphs with $\Delta = 3$ may require a special treatment. In [2], Bodlaender *et al.* proposed the following conjecture.

Conjecture 1.1 *For any outerplanar graph G of maximum degree Δ , $\lambda(G) \leq \Delta + 2$.*

Bodlaender *et al.* [2] themselves proved that $\lambda(G) \leq \Delta + 8$ for any outerplanar graph G and $\lambda(G) \leq \Delta + 6$ for any triangulated outerplanar graph. Calamoneri and Petreschi [6] proved that Conjecture 1.1 is true for any outerplanar graph with $\Delta \geq 8$. At the same time, as a corollary of their results on circular distance two labelings, Liu and Zhu [14] proved that this conjecture is true for outerplanar graphs with $\Delta \geq 15$. In [6], Calamoneri and Petreschi also proved that $\lambda \leq \Delta + 1$ for any triangulated outerplanar graph with $\Delta \geq 8$. Meanwhile, they gave [6] a counterexample to show that Conjecture 1.1 is false when $\Delta = 3$, indicating again that the case of maximum degree three may require a special treatment. Moreover, for any outerplanar graph G with $\Delta = 3$, they proved [6] that $\lambda(G) \leq \Delta + 5$ and $\lambda(G) \leq \Delta + 4$ if in addition G is triangle-free. Motivated by these results they asked [6] the following question.

Question 1.2 *Is the upper bound $\lambda(G) \leq \Delta + 5$ tight for outerplanar graphs with $\Delta = 3$?*

Although empirical results [4] suggest that $\Delta + 5$ may be improved for some sample outerplanar graphs with $\Delta = 3$, to our best knowledge the question above is still open, and no sharp upper bound on $\lambda(G)$ is known for all outerplanar graphs with $\Delta = 3$.

In this paper we answer the question above by proving that $\lambda(G) \leq \Delta + 3 = 6$ for every outerplanar graph with $\Delta = 3$. Moreover, we prove that this bound can be achieved by infinitely many outerplanar graphs with $\Delta = 3$. This extends the single outerplanar graph with $\Delta = 3$ and $\lambda = 6$ given in [6] to an infinite family of such extremal graphs. A typical member in this family is the graph $G(l)$ such that $l \geq 4$ is not a multiple of 3, where $G(l)$, depicted in Figure 1, is defined as follows. (Note that $G(4)$ is the graph in [6, Fig. 8].)

Definition 1.3 Let $l \geq 3$ be an integer. Define $G(l)$ to be the graph with vertex set $\{u, v, x_1, x_2, \dots, x_l, y_1, y_2, \dots, y_l\}$ and edge set $\{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq l-1\} \cup \{ux_1, uy_1, vx_l, vy_l\} \cup \{x_i y_i : 1 \leq i \leq l\}$.

The main result of this paper is as follows.

Theorem 1.4 *For any outerplanar graph G with maximum degree $\Delta = 3$, we have $\lambda(G) \leq 6$. Moreover, for any integer $l \geq 4$ which is not a multiple of 3, the outerplanar graph $G(l)$ as shown in Figure 1 satisfies $\lambda(G(l)) = 6$.*

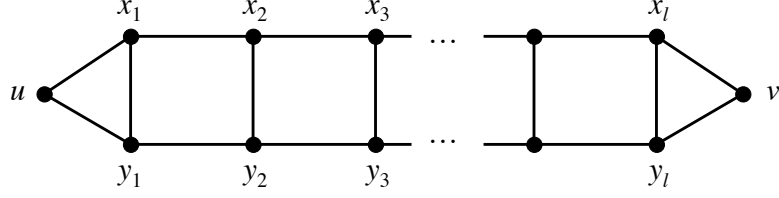


Figure 1: The outerplanar graph $G(l)$.

It is clear that, for any graph G , $l_{1,1}(G) + 1$ is equal to the chromatic number $\chi(G^2)$ of the square of G . Wegner [18] conjectured that, for any planar graph G , $\chi(G^2)$ is bounded from above by 7 if $\Delta = 3$, by $\Delta + 5$ if $4 \leq \Delta \leq 7$, and by $\lceil 3\Delta/2 \rceil + 1$ if $\Delta \geq 8$. The aforementioned result of Molloy and Salavatipour [15], which can be restated as $\chi(G^2) \leq \lceil 5\Delta/3 \rceil + 78$, is the best known result on this conjecture for general Δ . Thomassen [17] proved that Wegner's conjecture is true for planar graphs with $\Delta = 3$. Since $\chi(G^2) - 1 = l_{1,1}(G) \leq l(G)$, Theorem 1.4 can be viewed as a refinement of Thomassen's result for outerplanar graphs (but not general planar graphs) with $\Delta = 3$.

In order to prove Theorem 1.4, we will introduce two extension techniques in Sections 4 and 5, respectively. (See Theorems 4.6 and 5.4.) Developed from graph coloring theory, they will play a key role in our proof (in Section 6) of the upper bound $\lambda(G) \leq 6$. In Section 3 we will prove $\lambda(G(l)) \geq 6$, which together with the upper bound implies the second statement in Theorem 1.4.

We follow [3] for terminology and notation on graphs. All graphs considered are simple and undirected. The neighborhood of v in a graph G is denoted by $N(v)$, and the set of vertices at distance two from v in G is denoted by $N_2(v)$. A graph is *outerplanar* if it can be embedded in the plane in such a way that all the vertices lie on the boundary of the same face called the *outer face*. When an outerplanar graph G is drawn in this way in a plane, we call it an *outer plane graph*. The boundary of a face F of an outer plane graph, denoted by ∂F , can be regarded as the subgraph induced by its vertices. If F is circular with vertices x_1, x_2, \dots, x_k in order, then $\partial F = x_1 x_2 \dots x_k x_1$ is a k -cycle and we call F a k -face. (A k -cycle is a cycle of length k .) Two faces of an outer plane graph are *intersecting* if they share at least one common vertex.

If H is a subgraph of G and $f : V(G) \rightarrow [0, k] = \{1, 2, \dots, k\}$ is a k - $L(2, 1)$ -labeling of G , then we define $f|_H : V(H) \rightarrow [0, k]$ to be the restriction of f to $V(H)$; that is, $f|_H(u) = f(u)$ for each $u \in V(H)$. Clearly, $f|_H$ is a k - $L(2, 1)$ -labeling of H . Conversely, if f is a 6 - $L(2, 1)$ -labeling of H and f_1 a 6 - $L(2, 1)$ -labeling of G such that $f_1|_H$ is identical to f , then we say that f can be *extended* to f_1 . In this case, we will simply use f instead of f_1 to denote the extended labeling.

2 Preliminaries

A graph G is said to be the 2 -sum of its subgraphs H_1 and H_2 , written $G = H_1 \oplus_2 H_2$, if $V(G) = V(H_1) \cup V(H_2)$, $E(G) = E(H_1) \cup E(H_2)$, $|V(H_1) \cap V(H_2)| = 2$ and $|E(H_1) \cap E(H_2)| = 1$.

Lemma 2.1 *Suppose G is a 2-connected outer plane graph with $\Delta = 3$, and F_1 and F_2 are two 3-faces of G . Then F_1 and F_2 are intersecting if and only if G is isomorphic to $F_1 \oplus_2 F_2$.*

Proof. Assume $\partial F_1 = u_1 u_2 u_3$ and $\partial F_2 = v_1 v_2 v_3$. Suppose F_1 and F_2 are intersecting and let, say, $u_1 = v_1$. Since $\Delta = 3$, we have $u_2 = v_2$ or $u_2 = v_3$. Without loss of generality we may assume $u_2 = v_2$. Then $d(u_1) = d(u_2) = 3$ and $d(u_3) = d(v_3) = 2$. Since G is an outer plane graph, $u_3 v_3 \notin E(G)$. It follows from the 2-connectivity of G that $G = F_1 \oplus_2 F_2$. Obviously, if G is isomorphic to $F_1 \oplus_2 F_2$, then F_1 and F_2 are intersecting. ■

Lemma 2.2 *Suppose G is an outerplanar graph with $\Delta = 3$ and v is a vertex of G with $d(v) = 2$. Let $P = v_1 v_2 \dots v_q$ be a path such that $V(G) \cap V(P) = \emptyset$. Let G' denote the graph obtained from G and P by identifying v with v_1 . Then G admits a $6-L(2, 1)$ -labeling if and only if G' admits a $6-L(2, 1)$ -labeling.*

Proof. If g is a $6-L(2, 1)$ -labeling of G' , then $g|_G$ is a $6-L(2, 1)$ -labeling of G . Conversely, let f be a $6-L(2, 1)$ -labeling of G . Let u_1 and u_2 be two neighbors of v in G . Denote $a_1 = f(u_1)$, $a_2 = f(u_2)$ and $b = f(v)$. We extend f to the vertices of P as follows. The vertex v_2 is assigned $f(v_2) \in [0, 6] \setminus \{a_1, a_2, b, b-1, b+1\}$, and for $3 \leq j \leq q$, v_j is assigned $f(v_j) \in [0, 6] \setminus \{f(v_{j-2}), f(v_{j-1}) - 1, f(v_{j-1}) + 1\}$. One can verify that this extension is possible and it defines a $6-L(2, 1)$ -labeling of G' . ■

In Theorems 4.6 and 5.4, we will develop two extension techniques under which a given $6-L(2, 1)$ -labeling of a subgraph of G can be extended to a $6-L(2, 1)$ -labeling of G . To this end we define three classes of extendable $6-L(2, 1)$ -labelings as follows. (See Figure 2 for an illustration.)

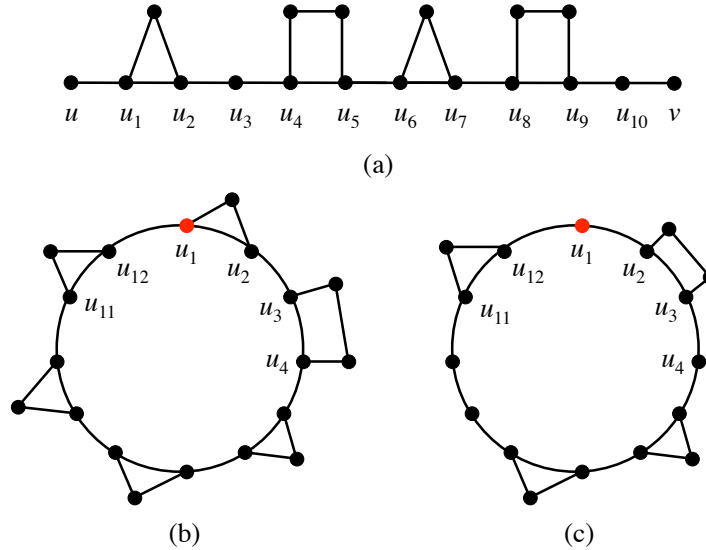


Figure 2: An illustration of Definitions 2.3 and 2.4: (a) A graph in $\mathcal{H}(P)$ with $l = 10$, $t = 4$, $i_1 = 1$, $i_2 = 4$, $i_3 = 6$ and $i_4 = 8$; (b) a graph in $\mathcal{H}(C)$ with $l = 12$ and $t = 6$, where C has starting vertex u_1 ; (c) a graph in $\mathcal{F}(C)$ with $l = 12$ and $t = 5$, where C has starting vertex u_1 . Note that the graph in (b) is not a member of $\mathcal{F}(C)$.

Definition 2.3 Given a path $P = uu_1u_2 \dots u_lv$ with $l \geq 2$, we define a family of outer plane graphs $\mathcal{H}(P)$ as follows: $G \in \mathcal{H}(P)$ if and only if there exist i_1, i_2, \dots, i_t with $1 \leq i_1 < i_2 < \dots < i_t < l$ and $i_{j+1} \geq i_j + 2$ for $j = 1, \dots, t-1$, such that G can be obtained from P by attaching t paths of length 2 or 3 to P and identifying the two end vertices of the j th path to u_{i_j} and u_{i_j+1} respectively, for $j = 1, \dots, t$.

A 6- $L(2,1)$ -labeling f of P is called a path-extendable 6- $L(2,1)$ -labeling if f can be extended to a 6- $L(2,1)$ -labeling of every $H \in \mathcal{H}(P)$.

Definition 2.4 Given a cycle $C = u_1u_2 \dots u_lu_1$ with starting vertex u_1 (where $l \geq 3$), we define two families of outer plane graphs, denoted by $\mathcal{K}(C)$ and $\mathcal{F}(C)$, as follows: $G \in \mathcal{K}(C)$ if and only if there exist i_1, i_2, \dots, i_t with $1 \leq i_1 < i_2 < \dots < i_t \leq l$ and $i_{j+1} \geq i_j + 2$ for $j = 1, \dots, t-1$, such that G can be obtained from C by attaching t paths of length 2 or 3 to C and identifying the two end vertices of the j th path to u_{i_j} and u_{i_j+1} respectively, for $j = 1, \dots, t$, with the subscripts of u 's modulo l .

The family $\mathcal{F}(C)$ ($\subseteq \mathcal{K}(C)$) is defined in exactly the same way as $\mathcal{K}(C)$ except that in addition we require $i_1 \geq 2$ and $i_t < l$.

A 6- $L(2,1)$ -labeling f of C is defined to be a cycle-extendable 6- $L(2,1)$ -labeling of type 1 (respectively, type 2) if f can be extended to a 6- $L(2,1)$ -labeling of every $H \in \mathcal{K}(C)$ (respectively, $H \in \mathcal{F}(C)$).

We emphasize that $\mathcal{K}(C)$ and $\mathcal{F}(C)$ depend on the starting vertex u_1 of C , and that in our subsequent discussion u_1 should be clear from the context.

Path-extendable labelings will be used in extension technique 1 while cycle-extendable labelings in extension technique 2. Not every 6- $L(2,1)$ -labeling of P is extendable. For example, if a 6- $L(2,1)$ -labeling f of P satisfies $(f(u_i), f(u_{i+1}), f(u_{i+2}), f(u_{i+3})) = (4, 1, 3, 0)$ for some $1 \leq i \leq l-3$, then f is not extendable. In fact, if H is obtained from P by identifying the end vertices of a path Q of length 3 to u_{i+1} and u_{i+2} respectively, then $H \in \mathcal{H}(P)$ but f cannot be extended to a 6- $L(2,1)$ -labeling of H , because we cannot assign labels from $[0, 6]$ to the two middle vertices of Q without violating the $L(2,1)$ -condition. Similarly, if f is a 6- $L(2,1)$ -labeling of P such that $(f(x_i), f(x_{i+1}), f(x_{i+2}), f(x_{i+3})) = (0, 2, 4, 6), (4, 1, 6, 3), (5, 1, 4, 6)$ or $(6, 1, 4, 2)$ for some $1 \leq i \leq l-3$, then f is not extendable.

For a 6- $L(2,1)$ -labeling f of G , if $f(u) \in \{1, 3, 5\}$ for a vertex u , then there are two possible labels from $\{0, 2, 4, 6\}$ that can be assigned to a neighbor v of u such that $|f(u) - f(v)| \geq 2$. These two labels are called the *available neighbor labels* in $\{0, 2, 4, 6\}$ for v with respect to $f(u)$ (or available neighbor labels in $\{0, 2, 4, 6\}$ for $f(u)$). Similarly, there are two available neighbor labels in $\{1, 3, 5\}$ for a label $f(u) \in \{0, 6\}$ and one available neighbor label in $\{1, 3, 5\}$ for a label $f(u) \in \{2, 4\}$. The next lemma is straightforward.

Lemma 2.5 Given a 6- $L(2,1)$ -labeling f of an outer plane graph G with $\Delta = 3$, if L_x is the set of available neighbor labels in $\{0, 2, 4, 6\}$ for $x \in \{1, 3, 5\}$ and M_y the set of available neighbor labels in $\{1, 3, 5\}$ for $y \in \{0, 2, 4, 6\}$, then the following hold:

- (i) for any $a, b \in \{1, 3, 5\}$ with $a \neq b$, $|L_a \cup L_b| \geq 3$ and $|L_a \setminus L_b| \geq 1$;
- (ii) if $(a, b) = (1, 3)$ or $(3, 5)$, then $|L_a \cap L_b| = 1$;
- (iii) for each $a \in \{0, 2, 4, 6\}$, $|M_a| \geq 1$;
- (iv) for any $a, b \in \{0, 2, 4, 6\}$ with $a \neq b$, $|M_a \cup M_b| \geq 2$; moreover, $|M_a \cup M_b| \geq 3$ when $(a, b) \in \{(0, 6), (2, 6)\}$.

Given a path $u_1u_2 \dots u_{3k}$, we say that we label u_1, u_2, \dots, u_{3k} using *pattern abc* , or these vertices are labeled using pattern abc , if, for $0 \leq i \leq k-1$, u_{3i+1} , u_{3i+2} and u_{3i+3} are assigned a , b and c , respectively.

- Lemma 2.6** (i) Let $P = uu_1u_2 \dots u_{3k}v$ be a path and f a $6-L(2,1)$ -labeling of P . If f labels u_1, u_2, \dots, u_{3k} using pattern abc , where $\{a, b, c\} \subset \{0, 2, 4, 6\}$, then f is a path-extendable $6-L(2,1)$ -labeling.
- (ii) Let $C = u_1u_2 \dots u_{3k}u_1$ be a cycle and f a $6-L(2,1)$ -labeling of C . If f labels u_1, u_2, \dots, u_{3k} using pattern abc , where $\{a, b, c\} \subset \{0, 2, 4, 6\}$, then f is a cycle-extendable $6-L(2,1)$ -labeling of type 1.
- (iii) Let $C = u_1u_2 \dots u_{3k}u_{3k+1}u_1$ be a cycle and f a $6-L(2,1)$ -labeling of C . If f labels $u_3, u_4, \dots, u_{3k}, u_{3k+1}, u_1$ using pattern abc , where $\{a, b, c\} \subset \{0, 2, 4, 6\}$, then f is a cycle-extendable $6-L(2,1)$ -labeling of type 2.

Proof. We prove (a) here. The proofs for (b) and (c) are similar. Let $H \in \mathcal{H}(P)$. If a vertex v of H is adjacent to two consecutive vertices of P , then v can be assigned the unique label in $\{0, 2, 4, 6\} \setminus \{a, b, c\}$. If two adjacent vertices u, v of H are adjacent to u_i, u_{i+1} respectively, then by Lemma 2.5, u and v can be assigned labels from $\{1, 3, 5\}$. Thus f can be extended to a $6-L(2,1)$ -labeling of H . Since this is true for any $H \in \mathcal{H}(P)$, f is extendable. ■

All $6-L(2,1)$ -labelings of a path P (or a cycle) in this paper fall into two categories for some $a, b, c \in \{0, 2, 4, 6\}$: either all vertices of P are labeled using pattern abc , or all vertices of P except at most three vertices at each end of P are labeled using pattern abc . In the latter case, by Lemma 2.6, we have the following lemma.

- Lemma 2.7** (i) Let $P = uv_1 \dots v_t u_1 u_2 \dots u_{3k} w_1 \dots w_s v$ be a path and f a $6-L(2,1)$ -labeling of P . Suppose $f|_{uv_1 \dots v_t u_1 u_2}$ and $f|_{u_{3k-1} u_{3k} w_1 \dots w_s v}$ are path-extendable $6-L(2,1)$ -labelings of $uv_1 \dots v_t u_1 u_2$ and $u_{3k-1} u_{3k} w_1 \dots w_s v$, respectively, where $t \leq 3$ and $s \leq 3$. If f labels u_1, u_2, \dots, u_{3k} using pattern abc , where $\{a, b, c\} \subset \{0, 2, 4, 6\}$, then f is a path-extendable $6-L(2,1)$ -labeling.
- (ii) Let $C = v_1 \dots v_t u_1 u_2 \dots u_{3k} v_1$ be a cycle and f a $6-L(2,1)$ -labeling of C . Suppose $f|_{u_{3k-1} u_{3k} v_1 \dots v_t u_1 u_2}$ is an extendable $6-L(2,1)$ -labeling of $u_{3k-1} u_{3k} v_1 \dots v_t u_1 u_2$, where $t \leq 3$. If f labels u_1, u_2, \dots, u_{3k} using pattern abc , where $\{a, b, c\} \subset \{0, 2, 4, 6\}$, then f is a cycle-extendable $6-L(2,1)$ -labeling of type 1.
- (iii) Let $C = v_1 \dots v_t u_1 u_2 \dots u_{3k+1} u_1 u_2$ be a cycle and f a $6-L(2,1)$ -labeling of C . Suppose $f|_{u_2 v_1 \dots v_t u_3 u_4}$ is a path-extendable $6-L(2,1)$ -labeling of $u_2 v_1 \dots v_t u_3 u_4$, where $t \leq 3$. If f labels $u_3, u_4, \dots, u_{3k+1}, u_1$ using pattern abc , where $\{a, b, c\} \subset \{0, 2, 4, 6\}$, then f is a cycle-extendable $6-L(2,1)$ -labeling of type 2.

3 The lower bound

In this section we prove the relatively easy part of Theorem 1.4, that is, $\lambda(G(l)) \geq 6$ if $l \geq 4$ is not a multiple of 3. Throughout this section we assume $l \geq 3$ and the vertices $u, v, x_1, \dots, x_l, y_1, \dots, y_l$ of $G(l)$ are as shown in Figure 1.

Suppose that $G(l)$ admits a $5-L(2,1)$ -labeling $f : V(G(l)) \rightarrow [0, 5]$. Let H_1 and H_2 be the subgraphs of $G(l)$ induced by $f^{-1}(\{0, 2, 4\})$ and $f^{-1}(\{1, 3, 5\})$ respectively. The roles of H_1 and H_2 are symmetric because assigning $5 - f(w)$ to $w \in V(G(l))$ yields another $5-L(2,1)$ -labeling of $G(l)$.

Lemma 3.1 Every component of H_1 or H_2 has at least two vertices.

Proof. By the symmetry between H_1 and H_2 , it suffices to prove this for any component H of H_1 . Suppose to the contrary that $V(H) = \{w\}$ for some $w \in V(G(l))$. Since each vertex of $G(l)$ has degree 2 or 3, we have $|N(w)| = 2$ or 3. Assume $N(w) = \{w_1, w_2, w_3\}$ first. If $f(w) = i$, then $i \in \{0, 2, 4\}$

and $f(w_1), f(w_2), f(w_3) \in \{1, 3, 5\}$. By the $L(2, 1)$ -condition, $f(w_1)$, $f(w_2)$ and $f(w_3)$ are distinct and $\{f(w_1), f(w_2), f(w_3)\} = \{1, 3, 5\}$. Thus there is at least one w_j such that $f(w_j) = i + 1$ or $i - 1$. This is a contradiction as w_j is adjacent to w .

Assume then $N(w) = \{w_1, w_2\}$. By the structure of $G(l)$, we may assume $w = u$, $w_1 = x_1$ and $w_2 = y_1$. Since H is a component of H_1 , we have $f(u) \in S_1$ and $f(x_1), f(y_1) \in S_2$, which implies $f(u) = 0$ and $\{f(x_1), f(y_1)\} = \{3, 5\}$. Without loss of generality we may assume $f(x_1) = 3$ and $f(y_1) = 5$. By the $L(2, 1)$ -condition we have $f(x_2) = 1$. This implies that y_2 cannot be assigned any label from $[0, 5]$ without violating the $L(2, 1)$ -condition, a contradiction again. ■

Lemma 3.2 *Let H be a component of H_1 or H_2 . Then the following hold:*

- (i) *H contains no 4-cycle $x_i y_i y_{i+1} x_{i+1} x_i$, where $1 \leq i \leq l$;*
- (ii) *H cannot contain a 3-vertex and all its neighbors;*
- (iii) *if l is not a multiple of 3, then H cannot contain any cycle; if l is a multiple of 3, then either H itself is a 3-cycle or it does not contain any cycle.*

Proof. (i) This follows immediately from the $L(2, 1)$ -condition.

(ii) Suppose to the contrary that H contains a degree-three vertex w and its neighbors w_1, w_2, w_3 . Let $f(w) = j \in S_i$, where $i = 1$ or 2 . Then $f(w_1), f(w_2), f(w_3) \in S_i \setminus \{j\}$ and hence there exist $s, t \in \{1, 2, 3\}$, $s \neq t$ such that $f(w_s) = f(w_t)$. However, this violates the $L(2, 1)$ -condition.

(iii) Since the roles of H_1 and H_2 are symmetric, it suffices to prove the results for H_1 . Suppose that a component H of H_1 contains a cycle C . If $|V(C)| \geq 4$, then H contains $x_i, x_{i+1}, y_{i+1}, y_i$ for some i , contrary to (i). Thus $|V(C)| = 3$ and so by symmetry we may assume $C = ux_1y_1u$.

Consider $f(u) = 0$ first. In this case we may assume $f(x_1) = 2$ and $f(y_1) = 4$ by symmetry. Then $f(x_2) = 5$, $f(y_2) = 1$, $f(x_{3k}) = 0$, $f(y_{3k}) = 3$, $f(x_{3k+1}) = 2$, $f(y_{3k+1}) = 5$, $f(x_{3k+2}) = 4$ and $f(y_{3k+2}) = 1$ for $k \geq 1$. Thus $H = C$ is a 3-cycle and moreover v cannot be assigned any label from $[0, 5]$ unless 3 divides l .

In the case when $f(u) = 2$, we may assume $f(x_1) = 0$ and $f(y_1) = 4$ by symmetry. Then $f(x_2) = 3$ or 5 , and $f(y_2) = 1$. When $f(x_2) = 3$, we have $f(x_3) = 5$ and y_3 cannot be assigned any label from $[0, 5]$. When $f(x_2) = 5$, we have $f(x_3) = 3$ or 2 , and y_3 cannot be assigned any label from $[0, 5]$.

In the case when $f(u) = 4$, we may assume $f(x_1) = 0$ and $f(y_1) = 2$ by symmetry. Then $f(x_2) = 3$ and $f(y_2) = 5$. This implies that x_3 must be assigned 1 and y_3 cannot be assigned any label from $[0, 5]$. ■

A component H of H_i is said to be a *path component* if H is a path, where $i = 1, 2$. We say that a path component H contains a path P if $V(P) \subseteq V(H)$.

Lemma 3.3 *Let H be a path component of H_i , where $i = 1, 2$. Then an end vertex w of H with $d(w) = 3$ must be assigned 0 if $i = 1$ and 5 if $i = 2$.*

Proof. Since the roles of H_1 and H_2 are symmetric, it suffices to prove the result for $i = 1$. By Lemmas 3.1, the length of H is greater than 2. Assume $H = w_1 \dots w_l$ with $d(w_1) = 3$. Let $N(w_1) = \{w_2, z_1, z_2\}$. Then $z_1, z_2 \notin V(H)$ and $f(z_1), f(z_2) \in \{1, 3, 5\}$. If $f(w_1) = a \neq 0$, then $f(z_j) = a - 1$ or $a + 1$ for some $j = 1, 2$, which violates the $L(2, 1)$ -condition. ■

Lemma 3.4 *Let H be a path component of H_1 or H_2 . Then the following hold:*

- (i) *H contains no 3-path $x_i x_{i+1} y_{i+1} y_{i+2}$;*

- (ii) if H contains a 2-path $x_i y_i y_{i+1}$ (or $y_i x_i x_{i+1}$) such that x_i is an end vertex of a path component of H_j for $j = 1, 2$, then $i = 2$;
- (iii) if H contains a 2-path $x_i y_i y_{i-1}$ (or $y_i x_i x_{i-1}$) such that x_i is an end vertex of a path component of H_j for $j = 1, 2$, then $i = l - 1$.

Proof. Since the roles of H_1 and H_2 are symmetric, we may assume that H is a path component of H_1 .

(i) Suppose to the contrary that H contains the 3-path $x_i x_{i+1} y_{i+1} y_{i+2}$. By Lemma 3.2, $y_i, x_{i+2} \in V(H_2)$. By Lemma 3.3, $f(y_i) = f(x_{i+2}) = 5$. So $\{f(x_i), f(x_{i+1}), f(y_{i+1}), f(y_{i+2})\} = \{0, 2\}$, which violates the $L(2, 1)$ -condition.

(ii) Suppose that H contains such a 2-path $x_i y_i y_{i+1}$. If $i \geq 2$, then $x_{i-1} \in V(H_2)$ by (i). Hence $x_{i+1} \in V(H_2)$ and $y_{i-1} \in V(H_2)$ by Lemma 3.2. It follows that $f(x_{i+1}) = 5$, $f(x_i) = 0$, $f(x_{i-1}) = 3$, $f(y_{i-1}) = 1$, $f(y_{i+1}) = 2$ and $f(y_i) = 4$. If $i \geq 3$, then by symmetry x_{i-2} must be assigned 5, but then y_{i-2} cannot be assigned any label in $[0, 5]$. If $i = 1$, then $f(x_1) = 0$, $f(y_1) = 2$, $f(y_2) = 4$ and $f(u) = 5$ by symmetry, but then x_2 cannot be assigned any label $[0, 5]$. Therefore, $i = 2$.

(iii) The proof is similar to that of (ii). ■

Lemma 3.5 *Let H be a component of H_1 or H_2 . Then H is one of the following:*

- (i) a 3-cycle;
- (ii) the path $x_3 x_4 \dots x_l$ (or $y_3 y_4 \dots y_l$);
- (iii) the path $x_3 x_4 \dots x_{l-1} y_{l-1}$ (or $y_3 y_4 \dots y_{l-1} x_{l-1}$);
- (iv) the path $x_2 y_2 \dots y_{l-2}$ (or $y_2 x_2 \dots x_{l-2}$);
- (v) the path $u x_1 x_2 \dots x_l$ (or $u y_1 y_2 \dots y_l$);
- (vi) the path $y_1 y_2 \dots y_l v$ (or $x_1 x_2 \dots x_l v$);
- (vii) the path $x_2 y_2 \dots y_l v$ (or $y_2 x_2 \dots x_l v$).

Proof. By symmetry, we may assume that H is a component of H_1 and $x_i \in V(H)$ with minimum subscript i . By Lemma 3.1, $x_{i+1} \in V(H)$ or $y_i \in V(H)$.

Assume first that $x_{i+1} \in V(H)$ and $y_i \notin V(H)$. Let $x_j \in V(H)$ be such that j is maximum. By Lemma 3.4, $x_i, x_{i+1}, \dots, x_j \in V(H)$. If $i = 1$, then $y_1 \in V(H_2)$ by Lemma 3.4. By symmetry, $j = l$. By Lemma 3.2, the vertices of H are assigned 024024...0240 or 042042...0420 successively. We thus conclude that H is the path $u x_1 x_2 y_2 \dots x_l$ or $x_1 x_2 \dots x_l v$ and hence (v) or (vi) holds. If $i \geq 2$ and $i \neq 3$, then $x_{i-1} \in V(H_2)$. Since $y_i \in V(H_2)$, $y_{i-1} \notin V(H_2)$ by Lemma 3.4. Thus $y_{i-1} \in V(H_1)$. It follows that x_{i-1} is an end vertex of a path of H_2 and so is y_i . By Lemma 3.3, x_{i-1} and y_i are assigned 5, contradicting the $L(2, 1)$ -condition. Thus, assume that $i = 3$. By Lemma 3.4, $j = l - 1$ or l . If $j = l - 1$, by Lemma 3.4, $y_{l-1} \in V(H)$, $y_l, y_{l-2} \in V(H_2)$. We thus conclude that (iii) holds. If $j = l$, then (ii) holds.

Next we assume that $x_{i+1} \notin V(H)$ and $y_i \in V(H)$. If $i = 1$, then $x_2, y_2 \in V(H_2)$ by Lemma 3.4. It follows that H is a 3-cycle and (i) holds. So we assume $i \geq 2$. If $i = 2$, then $x_1 \in V(H_2)$ and $y_1 \in V(H_2)$ by Lemma 3.4. Thus $x_3 \in V(H)$ or $y_3 \in V(H)$. By symmetry, we may assume $x_3 \in V(H)$. By Lemma 3.4, we may assume $H = y_2 x_2 \dots x_j$ such that j is maximum. By Lemma 3.4, $i = 2$ and $j \geq l - 2$. Let $j = l - 2$. In this case, (iv) holds. If $j = l - 1$, then $y_{l-1} \in V(H_2)$ by Lemma 3.4. Thus, $y_l, x_l \in V(H_2)$, contradicting Lemma 3.4. If $j = l$, by Lemma 3.2, $y_3, y_4, \dots, y_l \in V(H_2)$, which form a path. Note that the vertices of H are assigned 024024...0240 or 042042...0420 sequentially,

and y_3, y_4, \dots, y_l should be assigned 531531...5315 or 513513...5135 sequentially, which implies $v \in V(H)$. We conclude that (ii) holds.

Finally, we assume that $x_{i+1} \in V(H)$ and $y_i \in V(H)$. Then $i = 2$ by Lemma 3.4. Let $x_3, \dots, x_j \in V(H)$ such that j is the maximum subscript. Then $x_{j+1}, y_{j+1}, y_j, \dots, y_3 \in V(H_2)$. By Lemma 3.4, $j = l - 2, l - 1, l$. If $j = l - 1$, then $y_3, \dots, y_l, x_l, v \in V(H_2)$, which implies that v, y_l, x_l, x_{l-1} cannot be assigned labels from $\{1, 3, 5\}$ without violating the $L(2, 1)$ -condition, a contradiction. Thus, $j = l - 2$ or l , which means that (iv) or (vii) holds. ■

Theorem 3.6 *Let $l \geq 4$ be an integer which is not a multiple of 3. Then $\lambda(G(l)) \geq 6$.*

Proof. Suppose to the contrary that $G(l)$ admits a $5-L(2, 1)$ -labeling f . Recall that the vertices $u, v, x_1, \dots, x_l, y_1, \dots, y_l$ of $G(l)$ are as shown in Figure 1. By symmetry, we may assume that H is a component of H_1 containing u . Then H is as in (i) or (v) of Lemma 3.5. If H is as in (i), then by Lemmas 3.4 and 3.5, the path in (ii) or (iii) is a component K of H_1 . If K is as in (ii), then the path in (vii) is a component of H_2 . By Lemma 3.3, the vertices of K are assigned 024024...0240 or 042042...0420 sequentially. By Lemma 3.5, $l - 2 = 3k + 1$ and l is a multiple of 3, a contradiction. The proof is similar in the case when K is as in (iii). If H is as in (v), then x_l and u must be assigned 0 by Lemma 3.3. Thus the vertices of $x_l, x_{l-1}, \dots, x_1, u$ must be assigned 024...024, ..., 0240 or 042...042..., 0420 sequentially. By Lemma 3.5, $l + 1 = 3k + 1$ and l is a multiple of 3, a contradiction again. ■

4 Extension technique 1

Notation 4.1 Let $C = u_1 u_2 \dots u_l u_1$ be a cycle of length $l \geq 4$, and let v_1 and v_2 be two additional vertices not on C . Define H be the graph obtained from C by adding the edges $u_1 v_1$ and $u_2 v_2$. Denote $P = v_1 u_1 u_2 v_2$, which is a path of H . Let H_1 denote the subgraph of H induced by $\{v_1, v_2, u_1, u_2, u_3, u_l\}$.

Throughout this section, H, P and H_1 are as above, and f is a fixed path-extendable $6-L(2, 1)$ -labeling of P .

The main result in this section, Theorem 4.6, states that any given path-extendable $6-L(2, 1)$ -labeling of P can be extended to a $6-L(2, 1)$ -labeling of H . To establish this result we need to prove a few lemmas first.

Lemma 4.2 *Suppose $\{f(u_1), f(u_2)\} \cap \{1, 3, 5\} \neq \emptyset$ and f can be extended to a $6-L(2, 1)$ -labeling of H_1 such that $f(u_3), f(u_l) \in \{0, 2, 4, 6\}$. Suppose further that $f(u_3) = f(u_l)$ if and only if $l \equiv 0 \pmod{3}$. Then f can be extended to a $6-L(2, 1)$ -labeling f_1 of H such that $f_1|_{C - \{u_1 u_2\}}$ is a path-extendable $6-L(2, 1)$ -labeling.*

Proof. Denote $f(u_3) = a$ and $f(u_l) = b$. Our assumption means $|\{f(u_1), f(u_2)\} \cap \{1, 3, 5\}| = 1$ or 2. Let us first consider the latter case, that is, $\{f(u_1), f(u_2)\} \subseteq \{1, 3, 5\}$. Since $\{0, 2, 4, 6\} \setminus \{a, b\} \neq \emptyset$, we can choose $c \in \{0, 2, 4, 6\} \setminus \{a, b\}$. If $l \equiv 2 \pmod{3}$, then we label u_4, u_5, \dots, u_l using pattern cba , and label u_{l-1} by c . In the case $l \equiv 1 \pmod{3}$, if $l = 4$, there is nothing to prove; if $l \geq 5$, then we label u_4, \dots, u_{l-1} using pattern bca . In the case $l \equiv 0 \pmod{3}$, we have $f(u_3) = f(u_l) = a$ by our assumption. Choose $b, c \in \{0, 2, 4, 6\} \setminus \{a\}$ such that $b \neq c$. We label u_4, u_5 by b, c respectively, and u_6, u_7, \dots, u_{l-1} using pattern abc .

Assume $|\{f(u_1), f(u_2)\} \cap \{1, 3, 5\}| = 1$ from now on. By symmetry we may assume $f(u_1) \in \{1, 3, 5\}$ and $f(u_2) \in \{0, 2, 4, 6\}$.

Consider the case $l \equiv 0 \pmod{3}$ first. In this case, $f(u_3) = f(u_l) = a$. Denote $b = f(u_2)$ and take $c \in \{0, 2, 4, 6\} \setminus \{a, b\}$. We label u_4, u_5 by c, b respectively and u_6, \dots, u_{l-1} using pattern acb .

Now we consider the case $l \equiv 2 \pmod{3}$. Since $f(u_1)$ is a common available neighbor label in $\{1, 3, 5\}$ for $f(u_2)$ and $f(u_l)$, $(f(u_2), f(u_l)) \in \{(6, 0), (0, 6), (0, 2), (2, 0), (4, 6), (6, 4)\}$. If $(f(u_2), f(u_l)) \in \{(0, 2), (2, 0), (4, 6), (6, 4)\}$, we label u_4 by $c \in \{0, 2, 4, 6\} \setminus \{a, b, f(u_2)\}$, and u_5, \dots, u_{l-1} using pattern bac . We are left with the case where $(f(u_2), f(u_l)) = (0, 6)$ or $(6, 0)$, for which $f(u_1) = 3$ and $f(v_1) \in \{1, 5\}$.

Suppose $(f(u_2), f(u_l)) = (0, 6)$. If $f(v_2) \neq 5$, then we re-assign 5 to u_3 , assign 1 to u_4 , and label u_5, u_6, \dots, u_{l-1} using pattern 602. Assume $f(v_2) = 5$. If $f(v_1) = 1$, then we re-assign 5 to u_l , 2 to u_{l-1} , 4 to u_{l-2} and label $u_{l-3}, u_{l-4}, \dots, u_3$ using pattern 024; if $f(v_1) = 5$, then we re-assign 1 to u_l , 6 to u_{l-1} , 2 to u_{l-2} and label $u_{l-3}, u_{l-4}, \dots, u_3$ using pattern 062.

Suppose $(f(u_2), f(u_l)) = (6, 0)$. If $f(v_2) \neq 1$, then we re-assign 1 to u_3 , assign 5 to u_4 , and label u_5, u_6, \dots, u_{l-1} using pattern 042. Assume $f(v_2) = 1$. In this case, since $f(u_1) = 3$, $f(v_1) \in \{1, 5\}$. If $f(v_1) = 5$, then we re-assign 2 to u_3 , assign 5 to u_4 and 1 to u_5 , and label u_6, u_7, \dots, u_l using pattern 640 when $l \geq 8$; we re-assign 2 to u_3 and 0 to u_5 , and assign 4 to u_4 when $l = 5$. If $f(v_2) = f(v_1) = 1$, then we re-assign 5 to u_l , assign 0 to u_{l-1} and 4 to u_{l-2} , and label $u_{l-3}, u_{l-4}, \dots, u_3$ using pattern 604.

Finally, in the case $l \equiv 1 \pmod{3}$, if $l = 4$, there is nothing to prove; if $l \geq 5$, then we label u_4, \dots, u_{l-1} using pattern $b f(u_2) a$.

In each possibility above, by Lemmas 2.6 and 2.7, we obtain a $6-L(2, 1)$ -labeling f_1 of H with the desired property. ■

Lemma 4.3 *If $l \equiv 0 \pmod{3}$ and $\{f(u_1), f(u_2)\} \cap \{1, 3, 5\} \neq \emptyset$, then f can be extended to a $6-L(2, 1)$ -labeling f_1 of H such that $f_1|_{C - \{u_1 u_2\}}$ is a path-extendable $6-L(2, 1)$ -labeling.*

Proof. We first prove:

Claim. If we can assign u_3 a label from $\{1, 3, 5\}$ and assign u_l a label from $\{0, 2, 4, 6\}$ such that they have no conflict with the existing labels, then f can be extended to a $6-L(2, 1)$ -labeling f_1 of H such that $f_1|_{C - \{u_1 u_2\}}$ is a path-extendable $6-L(2, 1)$ -labeling.

Proof of the Claim. Assume first that $f(u_1), f(u_2) \in \{1, 3, 5\}$. By the $L(2, 1)$ -condition, $f(u_3) \neq f(u_1)$. By Lemma 2.5 (a), $f(u_3)$ has an available neighbor label a in $\{0, 2, 4, 6\}$ which is not an available neighbor label for $f(u_1)$. It follows that $a \neq f(u_l)$. Let $c \neq a$ be an available neighbor label for $f(u_3)$, and let $b \in \{0, 2, 4, 6\} \setminus \{f(u_l), a, c\}$. We label u_4, u_5, \dots, u_l using pattern $ab f(u_l)$.

Now we assume $|\{f(u_1), f(u_2)\} \cap \{1, 3, 5\}| = 1$. Then $f(u_1) \in \{1, 3, 5\}$ or $f(u_2) \in \{1, 3, 5\}$.

Case 1: $f(u_1) \in \{1, 3, 5\}$.

Then $f(u_2) \in \{0, 2, 4, 6\}$. By our assumption, $f(u_3) \in \{1, 3, 5\}$. Since $f(u_1), f(u_3) \in \{1, 3, 5\}$, $f(u_2)$ is an available neighbor label for both $f(u_1)$ and $f(u_3)$. This leads to $f(u_2) = 6$ or 0 . Moreover, when $f(u_2) = 6$, $(f(u_1), f(u_3)) \in \{(3, 1), (1, 3)\}$; when $f(u_2) = 0$, $(f(u_1), f(u_3)) \in \{(3, 5), (5, 3)\}$. If $(f(u_1), f(u_3)) = (5, 3)$, then by assumption, $f(u_l) \in \{0, 2, 4, 6\}$, which implies $f(u_l) = 2$ and $f(v_1) \neq 2$. In this case, we label u_4, \dots, u_l using pattern 602. If $(f(u_1), f(u_3)) = (1, 3)$, then by assumption, $f(u_l) \in \{0, 2, 4, 6\}$, which implies $f(u_l) = 4$ and $f(v_1) \neq 4$. Thus, we label u_4, \dots, u_l using pattern 064.

Consider $(f(u_1), f(u_3)) = (3, 5)$. By our assumption, $f(u_l) \in \{0, 2, 4, 6\}$, which implies $f(u_l) = 6$ and $f(v_1) \neq 6$. If $f(v_2) \neq 6$, then we re-assign 6 to u_3 , and label u_4, \dots, u_l using pattern 206. If $f(v_2) = 6$, then we re-assign 4 to u_3 , and label u_4, \dots, u_l using pattern 206.

Consider $(f(u_1), f(u_3)) = (3, 1)$. If $f(v_2) \neq 0$, then we re-assign 0 to u_3 , and label u_4, \dots, u_l using pattern 460. If $f(v_2) = 0$, then we re-assign 2 to u_3 , and label u_4, \dots, u_l using pattern 460.

Case 2: $f(u_2) \in \{1, 3, 5\}$.

Then $f(u_1) \in \{0, 2, 4, 6\}$. If $f(u_1)$ is an available neighbor label in $\{0, 2, 4, 6\}$ for $f(u_3)$, let b be the other available neighbor label in $\{0, 2, 4, 6\}$ for $f(u_3)$, that is, $b \neq f(u_1)$. We label u_4, \dots, u_l using pattern

$f(u_1)ac$, where $a \in \{0, 2, 4, 6\} \setminus \{f(u_1), b\}$ and $c \in \{0, 2, 4, 6\} \setminus \{f(u_1), a\}$. In what follows we assume that

$$f(u_1) \text{ is not an available neighbor label for } f(u_3). \quad (1)$$

Assume $f(u_2) = 1$ first. Then $f(u_1) \in \{4, 6\}$. Assume first that $f(u_1) = 6$. If $f(v_2) \neq 3$, then we can re-assign 3 to u_3 . Thus $f(u_1)$ is an available neighbor label in $\{0, 2, 4, 6\}$ for $f(u_3)$, which contradicts (1). If $f(v_2) = 3$, then u_3 is assigned 5 by assumption. In this case, $f(v_1) \neq 4$ since f is a path extendable labeling of P . Thus, we re-assign 4 to u_3 , and label u_4, \dots, u_l using pattern 624.

Therefore, we may assume that $f(u_1) = 4$. Since $f(u_3) \in \{1, 3, 5\}$ and $f(u_2) = 1$, we have $f(u_3) \in \{3, 5\}$. Consider $f(u_3) = 3$. If $f(v_1) = 0$, then label u_4 by 5, u_5, \dots, u_{l-2} using pattern 264, and label u_{l-1} and u_l by 2, 6 respectively. If $f(v_1) \neq 0$, then label u_4, \dots, u_l using pattern 620. Consider $f(u_3) = 5$. In this case, label u_4 by 3 and u_5, \dots, u_{l-2} using pattern 6a4, where $a \in \{0, 2, 4, 6\} \setminus \{6, 4, f(v_1)\}$, and label u_{l-1} and u_l by 6 and a respectively.

Next we assume $f(u_2) = 3$. Then $f(u_1) \in \{0, 6\}$ and we can re-assign u_3 a label such that $f(u_1)$ is an available neighbor label in $\{0, 2, 4, 6\}$ for both $f(u_2)$ and $f(u_3)$, which contradicts (1).

Finally, we assume $f(u_2) = 5$. Then $f(u_1) \in \{0, 2\}$. We first assume that $f(u_1) = 0$. If $f(v_2) \neq 3$, then we can re-label u_3 by 3. Thus $f(u_1)$ is an available neighbor label in $\{0, 2, 4, 6\}$ for $f(u_3)$, which contradicts (1). Assume $f(v_2) = 3$. If $f(v_1) \neq 2$, then we re-label u_3 by 2 and label u_4, \dots, u_l using pattern 042; if $f(v_1) = 2$, then we re-label u_3 by 2 and label u_4, \dots, u_l using pattern 064. Thus, we assume that $f(u_1) = 2$. In the case when $f(u_3) = 1$, label u_4 by 3 and u_5, \dots, u_{l-2} using pattern 0a2, where $a \in \{0, 2, 4, 6\} \setminus \{0, 2, f(v_1)\}$, and label u_{l-1} and u_l by 0 and a respectively. In the case when $f(u_3) = 3$, if $f(v_1) \neq 0$, then label u_4 by 1 and u_5, \dots, u_{l-3} using pattern 402, and label u_{l-2} and u_{l-1} by 4 and 0 respectively; if $f(v_1) = 0$, then label u_4, \dots, u_l using pattern 046.

By Lemmas 2.6 and 2.7, in each possibility above, we obtain a 6- $L(2, 1)$ -labeling f_1 of H with the desired property. This completes the proof of the claim. \square

We are now ready to prove our lemma. Assume first that $\{f(u_1), f(u_2)\} \subset \{1, 3, 5\}$. By symmetry we may assume $(f(u_1), f(u_2)) \in \{(1, 3), (1, 5), (3, 5)\}$. If $\{f(v_1), f(v_2)\} \cap \{0, 2, 4, 6\} \neq \emptyset$, then one of u_3 and u_l can be assigned a label from $\{0, 2, 4, 6\}$ and the other a label from $\{1, 3, 5\}$. By Claim 1, f can be extended to a 6- $L(2, 1)$ -labeling f_1 of H such that $f_1|_{C-\{u_1, u_2\}}$ is an extendable 6- $L(2, 1)$ -labeling. So we may assume $f(v_1), f(v_2) \in \{1, 3, 5\}$. Since f is an extendable 6- $L(2, 1)$ -labeling of P , $(f(u_1), f(u_2)) \in \{(3, 5), (1, 3)\}$. Then $f(u_1)$ and $f(u_2)$ have a common available neighbor label c in $\{0, 2, 4, 6\}$. Thus both u_3 and u_l can be assigned c . By Lemma 4.2, f can be extended to a 6- $L(2, 1)$ -labeling f_1 of H such that $f_1|_{C-\{u_1, u_2\}}$ is an extendable 6- $L(2, 1)$ -labeling.

Next we assume $|\{f(u_1), f(u_2)\} \cap \{1, 3, 5\}| = 1$. By symmetry we may assume $f(u_1) \in \{1, 3, 5\}$ and $f(u_2) \in \{0, 2, 4, 6\}$. Then u_l can be assigned a label from $\{1, 3, 5\}$ and u_3 a label from $\{0, 2, 4, 6\} \setminus \{f(u_2), f(v_2)\}$. By Claim 1, f can be extended to a 6- $L(2, 1)$ -labeling f_1 of H such that $f_1|_{C-\{u_1, u_2\}}$ is a path-extendable 6- $L(2, 1)$ -labeling. \blacksquare

The next lemma can be easily verified. It will be used in the proof of Lemma 4.5.

Lemma 4.4 *Let $W = w_1w_2w_3w_4$ be a path. If $(a, b) \in \{(6, 0), (0, 6), (2, 6), (6, 2), (0, 4), (4, 0), (4, 2), (2, 4)\}$, then there is a path-extendable 6- $L(2, 1)$ -labeling f of W such that*

- (i) $f(w_1) = b$ and $f(w_4) = a$;
- (ii) $f(w_i) \in \{1, 3, 5\}$ for $i = 2, 3$;
- (iii) for every $H \in \mathcal{H}(W)$, each vertex of $V(H) \setminus V(W)$ can be assigned a label from $[0, 6] \setminus \{a, b, f(w_2), f(w_3)\}$ such that the resulting labeling is a path-extendable 6- $L(2, 1)$ -labeling.

Lemma 4.5 *If $|\{f(v_1), f(v_2), f(u_1), f(u_2)\} \cap \{1, 3, 5\}| \leq 1$, then f can be extended to a 6- $L(2, 1)$ -labeling f_1 of H such that $f_1|_{C-\{u_1 u_2\}}$ is a path-extendable 6- $L(2, 1)$ -labeling.*

Proof. We distinguish the following two cases.

Case 1. $|\{f(u_1), f(u_2)\} \cap \{1, 3, 5\}| = 1$.

We may assume $l \equiv 1$ or $2 \pmod{3}$ for otherwise the result is true by Lemma 4.3. By symmetry we may assume $f(u_1) \in \{1, 3, 5\}$. Then $f(v_1), f(v_2), f(u_2) \in \{0, 2, 4, 6\}$. Consider the case $l \equiv 2 \pmod{3}$ first. Since $f(v_1), f(u_2) \in \{0, 2, 4, 6\}$, $f(v_1)$ and $f(u_2)$ are two available neighbor labels for $f(u_1)$. By Lemma 2.5, there is a label $a \in \{1, 3, 5\} \setminus \{f(u_1)\}$ such that $f(v_2)$ is an available neighbor label for a . Assign a to u_l and $f(v_2)$ to u_{l-1} . Choose $b \in \{0, 2, 4, 6\} \setminus \{f(v_2), f(u_2)\}$. Assign b to u_3 and label u_4, \dots, u_{l-2} using pattern $f(v_2) f(u_2) b$.

Next assume $l \equiv 1 \pmod{3}$. The vertex u_l can be assigned a label $f(u_l) \in \{1, 3, 5\} \setminus \{f(u_1)\}$ such that $f(u_l)$ has an available neighbor label $a \in \{0, 2, 4, 6\} \setminus \{f(u_2), f(v_2)\}$. Assign a to u_{l-1} and u_{l-4} , $f(u_2)$ to u_{l-2} , $f(v_2)$ to u_{l-3} , and label u_{l-5}, \dots, u_3 using pattern $f(u_2) f(v_2) a$.

Case 2. $\{f(u_1), f(u_2)\} \cap \{1, 3, 5\} = \emptyset$.

Since $|\{f(v_1), f(v_2), f(u_1), f(u_2)\} \cap \{1, 3, 5\}| \leq 1$, $|\{f(v_1), f(v_2)\} \cap \{1, 3, 5\}| \leq 1$. We will consider three subcases: $l \equiv 0, 1, 2 \pmod{3}$. In the case when $l \equiv 0$, we will consider two subcases: $|\{f(v_1), f(v_2)\} \cap \{1, 3, 5\}| = 0$ or 1 . In the case when $l \not\equiv 0$, we do not consider any subcase explicitly. In the case when $l \equiv 0 \pmod{3}$ and $|\{f(v_1), f(v_2)\} \cap \{1, 3, 5\}| = 1$, we may assume without loss of generality that $f(v_1) \in \{1, 3, 5\}$ and $f(u_1), f(u_2), f(v_2) \in \{0, 2, 4, 6\}$.

Consider the case $l \equiv 0 \pmod{3}$. If $|\{f(v_1), f(v_2)\} \cap \{1, 3, 5\}| = 1$, choose $a \in \{0, 2, 4, 6\} \setminus \{f(u_1), f(u_2), f(v_2)\}$. Then assign a to u_3 and label u_4, u_5, \dots, u_l using pattern $f(u_1) f(u_2) a$.

If $|\{f(v_1), f(v_2)\} \cap \{1, 3, 5\}| = 0$, by symmetry we may assume $f(u_1) < f(u_2)$. If $(f(u_1), f(u_2)) \in \{(0, 6), (0, 2)\}$, then $f(u_1)$ and $f(u_2)$ have a common available neighbor label $a \in \{1, 3, 5\}$ such that u_3 can be assigned a , and u_4, u_5, \dots, u_l can be labeled using pattern $f(u_1) f(u_2) f(v_2)$. We now assume that $(f(u_1), f(u_2)) \in \{(4, 6), (2, 6)\}$. If $f(v_1) \neq f(v_2)$, then we assign u_3 by 3, u_4 by 1, u_5 by 6, label u_6, \dots, u_{l-1} using pattern $f(v_2) f(u_1) 6$ and assign u_l by $f(v_2)$. If $f(v_1) = f(v_2)$, let $a \in \{0, 2, 4, 6\} \setminus \{f(u_1), f(u_2), f(v_1)\}$. We assign u_3 by 3, u_4 by 1, u_5 by 6, label u_6, \dots, u_{l-1} using pattern $a f(u_1) 6$ and assign u_l by a . It remains to consider the case $(f(u_1), f(u_2)) \in \{(0, 4), (2, 4)\}$. In this case, since f is an extendable $L(2, 1)$ -labeling of P , we have $f(v_1) = f(v_2)$. In the case $(f(u_1), f(u_2)) = (0, 4)$, if $f(v_1) = f(v_2) = 6$, then label u_3, u_4 by 1, 3 respectively, label u_5, u_6, \dots, u_{l-3} using pattern 620, and label u_{l-2}, u_{l-1} by 6 and 2, respectively; if $f(v_1) = f(v_2) = 2$, then label u_3, u_4 by 1, 5 respectively, label u_5, u_6, \dots, u_{l-3} using pattern 260, and label u_{l-2}, u_{l-1} by 2 and 6, respectively. In the case $(f(u_1), f(u_2)) = (2, 4)$, if $f(v_1) = f(v_2) = 6$, then label u_3, u_4 by 0, 3 respectively, label u_5, u_6, \dots, u_{l-3} using pattern 602, and label u_{l-2}, u_{l-1} by 6 and 0, respectively; if $f(v_1) = f(v_2) = 0$, then label u_3, u_4 by 1, 5 respectively, label u_5, u_6, \dots, u_{l-3} using pattern 062, and label u_{l-2}, u_{l-1} by 0 and 6, respectively.

Consider the case $l \equiv 1 \pmod{3}$. If $(f(u_1), f(u_2)) \in \{(6, 0), (0, 6), (2, 6), (6, 2), (0, 4), (4, 0), (4, 2), (2, 4)\}$, then by Lemma 4.4 we have an extendable 6- $L(2, 1)$ -labeling of path $u_2 u_3 u_4 u_5$ such that $f(u_5) = f(u_1)$. If $f(v_1) \neq f(v_2)$, then label u_6, \dots, u_{l-2} using pattern $f(u_2) f(v_2) f(u_1)$, and label u_{l-1} and u_l by $f(u_2)$ and $f(v_2)$, respectively. If $f(v_1) = f(v_2)$, let $a \in \{0, 2, 4, 6\} \setminus \{f(v_2), f(u_1), f(u_2)\}$. In this case, label u_6, \dots, u_{l-2} using pattern $f(u_2) a f(u_1)$, and u_{l-1} and u_l by $f(u_2)$ and a , respectively. Thus we assume $(f(u_1), f(u_2)) \in \{(0, 2), (2, 0), (6, 4), (4, 6)\}$. By Lemma 4.4, we have an extendable 6- $L(2, 1)$ -labeling of path $u_2 u_3 u_4 u_5$ such that $f(u_5) = f(u_1)$. If $f(v_1) = f(v_2)$, let $a \in \{0, 2, 4, 6\} \setminus \{f(u_1), f(u_2), f(v_2)\}$. In this case, label u_l, \dots, u_8 using pattern $a f(u_2) f(u_1)$, and label u_7, u_6 by $a, f(u_2)$, respectively. If $f(v_1) \neq f(v_2)$, then label u_l, \dots, u_8 using pattern $f(v_2) f(u_2) f(u_1)$, and label u_7, u_6 by $f(v_2)$ and $f(u_2)$, respectively.

Finally, we consider the case $l \equiv 2 \pmod{3}$. Suppose that $(f(v_2), f(u_2)) \in \{(6, 0), (6, 2), (0, 4), (0, 6)\}$. If $f(v_1) \neq f(v_2)$, then by Lemma 4.4 we have an extendable 6- $L(2, 1)$ -labeling of path $u_2u_3u_4u_5$ such that $f(u_5) = f(v_2)$, and u_6, \dots, u_l can be labeled using pattern $f(u_1)f(u_2)f(v_2)$. If $f(v_1) = f(v_2)$, let $a \in \{0, 2, 4, 6\} \setminus \{f(u_1), f(u_2), f(v_2)\}$. Then, by Lemma 4.4, we have an extendable 6- $L(2, 1)$ -labeling of path $u_2u_3u_4u_5$ such that $f(u_5) = a$, and u_6, \dots, u_l can be labeled using pattern $f(u_1)f(u_2)a$.

Suppose then that $(f(v_2), f(u_2)) \in \{(4, 6), (2, 0)\}$. If $f(v_1) \neq f(v_2)$, then by Lemma 4.4 we have an extendable 6- $L(2, 1)$ -labeling of path $u_2u_3u_4u_5$ such that $f(u_5) = f(v_2)$, and u_6, \dots, u_l can be labeled using pattern $f(u_1)f(u_2)f(v_2)$. Assume $f(v_1) = f(v_2)$. Let $a \in \{0, 2, 4, 6\} \setminus \{f(v_2), f(u_1), f(u_2)\}$. If $(f(v_2), f(u_2), f(u_1)) \in \{(4, 6, 2), (2, 0, 4)\}$, then we have an extendable 6- $L(2, 1)$ -labeling of path $u_2u_3u_4u_5$ such that $f(u_5) = a$ by Lemma 4.4, and u_6, \dots, u_l can be labeled using pattern $f(u_1)f(u_2)a$. If $(f(v_2), f(u_2), f(u_1)) \in \{(4, 6, 0), (2, 0, 6)\}$, then by Lemma 4.4 we have an extendable 6- $L(2, 1)$ -labeling of path $u_2u_3u_4u_5$ such that $f(u_5) = f(v_2)$, and u_6, \dots, u_l can be labeled using pattern $f(u_1)f(u_2)a$.

Table 1: Partial labeling in the case when $l \equiv 2 \pmod{3}$

$(f(u_1), f(u_2), f(v_1), f(v_2))$	$(f(u_3), f(u_4), f(u_5), f(u_6), \dots, f(u_l))$
(4, 6, *, 2)	1, 5, 2, 4, 6, 2, ..., 4, 6, 2
(0, 6, *, 2)	1, 5, 2, 0, 6, 2, ..., 0, 6, 2
(6, 0, *, 4)	3, 1, 4, 6, 0, 4, ..., 6, 0, 4
(2, 0, *, 4)	3, 1, 4, 2, 0, 4, ..., 2, 0, 4
(0, 2, *, 4)	5, 1, 4, 0, 2, 4, ..., 0, 2, 4
(0, 4, *, 2)	1, 6, 2, 0, 4, 2, ..., 0, 4, 2
(6, 4, *, 2)	1, 5, 2, 6, 4, 2, ..., 6, 4, 2
(0, 4, *, 6)	1, 3, 6, 0, 4, 6, ..., 0, 4, 6
(2, 4, *, 6)	1, 3, 6, 2, 4, 6, ..., 2, 4, 6
(4, 2, *, 0)	5, 3, 0, 4, 2, 0, ..., 4, 2, 0
(6, 2, *, 0)	5, 3, 0, 4, 2, 0, ..., 4, 2, 0

In the remaining case where $(f(v_2), f(u_2)) \in \{(2, 6), (4, 0), (2, 4), (4, 2), (6, 4), (0, 2)\}$, we give a 6- $L(2, 1)$ -labeling in Table 1 when $f(v_1) \neq f(v_2)$ with one exception that $(f(u_1), f(u_2), f(v_2)) = (6, 2, 4)$. In this exceptional case, if $f(v_1) \neq 1$, then we label u_3, u_4, \dots, u_{l-3} using pattern 062, and u_{l-2}, u_{l-1}, u_l are labeled by 0, 3, 5, respectively; if $f(v_1) = 1$, then we label u_l, u_{l-1}, \dots, u_6 using pattern 426, and u_5, u_4, u_3 are labeled by 4, 0, 5, respectively. In Table 1, the labels in the first column are the given labels of u_1, u_2, v_1, v_2 , where * stands for a label from either $\{0, 2, 4, 6\}$ or $\{1, 3, 5\}$ as $|\{f(u_1), f(u_2), f(v_1), f(v_2)\} \cap \{1, 3, 5\}| \leq 1$. In the second column of Table 1, the first three labels are assigned to u_3, u_4 and u_5 , respectively, and the rest labels are assigned to u_6, \dots, u_l using the shown pattern. It remains to consider $f(v_1) = f(v_2)$. If $(f(v_2), f(u_2)) \in \{(2, 6), (4, 0)\}$, let $a \in \{0, 2, 4, 6\} \setminus \{f(u_1), f(u_2), f(v_2)\}$. By Lemma 4.4 we have an extendable 6- $L(2, 1)$ -labeling of path $u_2u_3u_4u_5$ such that $f(u_5) = a$ and u_6, \dots, u_l can be labeled using pattern $f(u_1)f(u_2)a$. If $(f(v_2), f(u_2)) \in \{(2, 4), (4, 2), (6, 4), (0, 2)\}$, let $a \in \{0, 2, 4, 6\} \setminus \{f(u_1), f(u_2), f(v_2)\}$. By Lemma 4.4 we have an extendable 6- $L(2, 1)$ -labeling of path $u_1u_lu_{l-1}u_{l-2}$ such that $f(u_{l-2}) = a$ and u_{l-3}, \dots, u_3 can be labeled using pattern $f(u_2)f(u_1)a$.

In each possibility above, by Lemmas 2.6 and 2.7, we obtain a 6- $L(2, 1)$ -labeling f_1 of H with the desired property. ■

Theorem 4.6 *Any path-extendable 6- $L(2, 1)$ -labeling f of P can be extended to a 6- $L(2, 1)$ -labeling f_1 of H such that $f_1|_{C-\{u_1u_2\}}$ is a path-extendable 6- $L(2, 1)$ -labeling of the path.*

Proof. By Lemma 4.5, we may assume $|\{f(v_1), f(v_2), f(u_1), f(u_2)\} \cap \{1, 3, 5\}| \geq 2$. Assume first $|\{f(v_1), f(v_2), f(u_1), f(u_2)\} \cap \{1, 3, 5\}| = 3$, so that $\{f(u_1), f(u_2)\} \cap \{1, 3, 5\} \neq \emptyset$. By Lemma 4.3, $l \equiv 1$

or 2 (mod 3). In each case we can label u_3, u_l by distinct $a, b \in \{0, 2, 4, 6\}$ respectively. By Lemma 4.2, f can be extended to a 6- $L(2, 1)$ -labeling f_1 of H such that $f_1|_{C-\{u_1 u_2\}}$ is an extendable 6- $L(2, 1)$ -labeling. It remains to consider the case $|\{f(v_1), f(v_2), f(u_1), f(u_2)\} \cap \{1, 3, 5\}| = 2$. We distinguish the following cases.

Case 1. $|\{f(u_1), f(u_2)\} \cap \{1, 3, 5\}| = 2$.

By Lemma 4.3, $l \equiv 1$ or $2 \pmod{3}$. We have $f(v_1), f(v_2) \in \{0, 2, 4, 6\}$. Note that each of $f(u_1)$ and $f(u_2)$ has two available neighbor labels in $\{0, 2, 4, 6\}$. Thus both u_3 and u_l can be assigned labels in $\{0, 2, 4, 6\}$. We claim that u_3 and u_l can be assigned labels in $\{0, 2, 4, 6\}$ such that $f(u_3) \neq f(u_l)$. Suppose otherwise. Then $f(u_1)$ and $f(u_2)$ have a common available neighbor label in $\{0, 2, 4, 6\}$. This implies that $(f(u_1), f(u_2)) \in \{(3, 5), (1, 3)\}$ and $f(v_1) \neq f(v_2)$. If $(f(u_1), f(u_2)) = (3, 5)$, then $f(v_1) = 6$ and $f(v_2) = 2$, which contradicts the assumption that f is an extendable $L(2, 1)$ -labeling of the path $v_1 u_1 u_2 v_2$, because $(2, 5, 3, 6)$ is not an extendable $L(2, 1)$ -labeling of this path. Similarly, if $(f(u_1), f(u_2)) = (1, 3)$, then $f(v_1) = 4$ and $f(v_2) = 0$, which contradicts the assumption that f is an extendable $L(2, 1)$ -labeling of the path $v_1 u_1 u_2 v_2$. Thus u_3 and u_l can be assigned labels in $\{0, 2, 4, 6\}$ such that $f(u_3) \neq f(u_l)$. By Lemma 4.2, f can be extended to a 6- $L(2, 1)$ -labeling f_1 of H such that $f_1|_{C-\{u_1 u_2\}}$ is an extendable 6- $L(2, 1)$ -labeling.

Case 2. $|\{f(u_1), f(u_2)\} \cap \{1, 3, 5\}| = 1$.

By Lemma 4.3, $l \equiv 1$ or $2 \pmod{3}$. By symmetry, we may assume $f(u_1) \in \{1, 3, 5\}$. Since $|\{f(u_1), f(u_2), f(v_1), f(v_2)\} \cap \{1, 3, 5\}| = 2$, we have $|\{f(v_1), f(v_2)\} \cap \{1, 3, 5\}| = 1$. We claim that $f(v_1) \in \{1, 3, 5\}$. Suppose otherwise. Then $f(v_2) \in \{1, 3, 5\}$. In the case $f(u_1) = 1$, we have $f(u_2) = 6$, $f(v_2) = 3$ and $f(v_1) = 4$, which implies that f is not an extendable 6- $L(2, 1)$ -labeling of P , a contradiction. Thus $f(v_1) \in \{1, 3, 5\}$ and $f(v_2) \in \{0, 2, 4, 6\}$. The cases where $f(u_1) \in \{3, 5\}$ can be dealt with similarly. Since $f(u_1)$ has two available neighbor labels in $\{0, 2, 4, 6\}$, namely $f(u_2)$ and a , u_l can be assigned a , and u_3 can be assigned $b \in \{0, 2, 4, 6\} \setminus \{f(u_2), f(v_2), a\}$. By Lemma 4.2, f can be extended to a 6- $L(2, 1)$ -labeling f_1 of H such that $f_1|_{C-\{u_1 u_2\}}$ is an extendable 6- $L(2, 1)$ -labeling.

Case 3. $\{f(u_1), f(u_2)\} \cap \{1, 3, 5\} = \emptyset$.

In this case, $f(v_1), f(v_2) \in \{1, 3, 5\}$. By symmetry, we may assume $f(u_2) < f(u_1)$. If $l \equiv 0 \pmod{3}$, let $a \in \{0, 2, 4, 6\} \setminus \{f(u_1), f(u_2)\}$ and label u_3 and u_l by a . By Lemma 4.2, f can be extended to a 6- $L(2, 1)$ -labeling f_1 of H such that $f_1|_{C-\{u_1 u_2\}}$ is an extendable 6- $L(2, 1)$ -labeling. Thus, $l \equiv 1$ or $2 \pmod{3}$.

First, we assume $l \equiv 1 \pmod{3}$. If $(f(u_1), f(u_2)) \in \{(6, 4), (6, 2), (6, 0), (4, 0)\}$, let $c \in \{0, 2, 4, 6\} \setminus \{f(u_1), f(u_2)\}$. We label u_3 by c and we have an extendable 6- $L(2, 1)$ -labeling of path $u_3 u_4 u_5 u_6$ such that $f(u_6) = f(u_2)$. If $l = 7$, then label u_7 by c . Otherwise, we can label u_7, \dots, u_{l-1} using pattern $c f(u_1) f(u_2)$, and label u_l by c . If $(f(u_1), f(u_2)) = (2, 0)$, then $f(v_1) = 5$ and $f(v_2) \in \{3, 5\}$. Thus u_l and u_{l-1} can be labeled 6 and 3, respectively, and u_3, \dots, u_{l-2} can be labeled using pattern 4 2 0. If $(f(u_1), f(u_2)) = (4, 2)$, then $f(v_1) = 1$ and $f(v_2) = 5$. Thus u_l, u_{l-1} and u_{l-2} can be assigned 6, 3 and 1, respectively, u_{l-3}, \dots, u_5 can be labeled using pattern 4 0 2, and u_4 and u_3 are assigned 4 and 0, respectively.

Next we assume $l \equiv 2 \pmod{3}$. By symmetry, we may assume $f(u_1) < f(u_2)$. Since $|\{f(v_1), f(v_2)\} \cap \{1, 3, 5\}| = 2$, $f(v_1) \neq f(v_2)$. If $(f(u_1), f(u_2)) \in \{(0, 2), (0, 4)\}$, then let $b = 6$. By Lemma 4.4, we have an extendable 6- $L(2, 1)$ -labeling of path $u_1 u_l u_{l-1} u_{l-2}$ such that $f(u_{l-2}) = b$ and $f(u_l) \neq f(v_1)$, while $u_{l-3}, u_{l-4}, \dots, u_3$ are labeled using pattern $f(u_2) f(u_1) b$. If $(f(u_1), f(u_2)) = (0, 6)$, then $(f(v_1), f(v_2)) \in \{(3, 1), (5, 1), (5, 3)\}$. Since f is an extendable $L(2, 1)$ -labeling of P , $(f(v_1), f(v_2)) \neq (5, 3)$. In each case when $(f(v_1), f(v_2)) = (3, 1)$ or $(5, 1)$, we label u_l, u_{l-1}, \dots, u_6 using pattern 4 6 0 and label u_5, u_4, u_3

by 4, 1, 3 respectively. If $(f(u_1), f(u_2)) \in \{(2, 6), (4, 6)\}$, then let $b = 0 \in \{0, 2, 4\} \setminus \{f(u_1), f(u_2)\}$. By Lemma 4.4, we have an extendable 6- $L(2, 1)$ -labeling of path $u_2 u_3 u_4 u_5$ such that $f(u_5) = 0 = b$ and $f(v_2) \neq f(u_3)$, while u_6, u_7, \dots, u_l are labeled using pattern $f(u_1) f(u_2) b$. It remains to consider the case when $(f(u_1), f(u_2)) = (2, 4)$. Then $f(v_1) = 5$ and $f(v_2) = 1$. We label u_3, u_4, u_5 by 0, 5, 1, respectively, and $u_6 \dots, u_l$ using pattern 406.

In each possibility above, by Lemmas 2.6 and 2.7, we obtain a 6- $L(2, 1)$ -labeling f_1 of H with the desired property. ■

5 Extension technique 2

Notation 5.1 Let $P = v_1 v_2 v_3$ be a path and $C = u_1 u_2 \dots u_l u_1$ a cycle, $l \geq 3$, such that $V(P) \cap V(C) = \emptyset$. Throughout this section, K is the graph obtained from P and C by adding the edge $u_2 v_2$ between P and C , and f is a given 6- $L(2, 1)$ -labeling of P .

Lemma 5.2 *If $l \equiv 0 \pmod{3}$, then f can be extended to a 6- $L(2, 1)$ -labeling f_1 of K such that $f_1|_C$ is a cycle-extendable 6- $L(2, 1)$ -labeling of type 2 in C .*

Proof. If $f(v_2) \in \{0, 2, 4, 6\}$, then choose $a \in \{0, 2, 4, 6\} \setminus \{f(v_1), f(v_2), f(v_3)\}$ and assign it to u_2 . Take $b, c \in \{0, 2, 4, 6\} \setminus \{a, f(v_2)\}$. We label u_1, u_3 by b, c respectively and u_4, u_5, \dots, u_l using pattern bac .

Assume $f(v_2) \in \{1, 3, 5\}$. Suppose first that $|\{f(v_1), f(v_3)\} \cap \{0, 2, 4, 6\}| \leq 1$. Since $f(v_2)$ has two available neighbor labels in $\{0, 2, 4, 6\}$, we assign its other available neighbor label a to u_2 . Similarly, Take $b, c \in \{0, 2, 4, 6\} \setminus \{a, f(v_2)\}$. We label u_1, u_3 by b, c respectively and u_4, u_5, \dots, u_l using pattern bac . Now suppose that $\{f(v_1), f(v_3)\} \in \{0, 2, 4, 6\}$. Assign u_2 a label from $\{1, 3, 5\} \setminus \{f(v_2)\}$. Then $f(u_2)$ has two available neighbor labels a, b in $\{0, 2, 4, 6\}$. We assign a and b to u_1 and u_3 , respectively. Choose $c \in \{0, 2, 4, 6\} \setminus \{a, b\}$ and label u_4, u_5, \dots, u_l using pattern acb .

In each possibility above, by Lemmas 2.6 and 2.7, we obtain a 6- $L(2, 1)$ -labeling f_1 of H with the desired property. ■

Lemma 5.3 *If $l \equiv 1 \pmod{3}$, then f can be extended to a 6- $L(2, 1)$ -labeling f_1 of K such that $f_1|_C$ is a cycle-extendable 6- $L(2, 1)$ -labeling of type 2.*

Proof. We first assume $f(v_2) \in \{1, 3, 5\}$ and $|\{f(v_1), f(v_3)\} \cap \{1, 3, 5\}| \leq 1$. In this case, there exists a label in $\{1, 3, 5\}$ which can be assigned to u_2 . There are two available neighbor labels $a, b \in \{0, 2, 4, 6\}$ for $f(u_2)$ such that u_1 and u_3 can be assigned a and b , respectively. Label u_4 by some $c \in \{0, 2, 4, 6\} \setminus \{a, b\}$ and u_5, \dots, u_l using pattern abc .

Next assume $f(v_1), f(v_2), f(v_3) \in \{1, 3, 5\}$. If $f(v_2) = 1$, then label u_2 by 6, u_1, u_l, u_{l-1} by 3, 0, 2, respectively, and u_{l-2}, \dots, u_3 using pattern 402; if $f(v_2) = 3$, then label u_2 by 0, u_1, u_l, u_{l-1} by 5, 3, 6, respectively, and u_{l-2}, \dots, u_3 using pattern 026; if $f(v_2) = 5$, then label u_2 by 0, u_1, u_l, u_{l-1} by 3, 1, 6, respectively, and u_{l-2}, \dots, u_3 using pattern 026.

Finally, we assume $f(v_2) \in \{0, 2, 4, 6\}$. We label u_2 by some $a \in \{0, 2, 4, 6\} \setminus \{f(v_1), f(v_2), f(v_3)\}$. If $a \in \{2, 4\}$ has only one available neighbor label $d \in \{1, 3, 5\}$, then we can assign d to u_1 ; if $a \in \{0, 6\}$, then we choose its available neighbor label $d = 3$ and assign 3 to u_1 . Moreover, d has another available neighbor label b in $\{0, 2, 4, 6\}$. Choose $c \in \{0, 2, 4, 6\} \setminus \{a, b, f(v_2)\}$. We label u_2, u_3, \dots, u_{l-2} using pattern cba , and u_{l-1}, u_l by c, b , respectively.

In each case above, by Lemmas 2.6 and 2.7, we obtain a 6- $L(2, 1)$ -labeling f_1 of K with the desired property. ■

Theorem 5.4 *f can be extended to a 6- $L(2, 1)$ -labeling f_1 of K such that $f_1|_C$ is a cycle-extendable 6- $L(2, 1)$ -labeling of type 2.*

Proof. By Lemmas 5.2 and 5.3, we are left with the case $l \equiv 2 \pmod{3}$. We first assume that $f(v_1), f(v_2), f(v_3) \in \{1, 3, 5\}$. If $f(v_2) = 1$, then we label $u_2, u_1, u_l, u_{l-1}, u_{l-2}$ by 6, 3, 1, 4, 0, respectively, and label u_{l-3}, \dots, u_3 using pattern 640. If $f(v_2) = 3$, then label $u_2, u_1, u_l, u_{l-1}, u_{l-2}$ by 6, 1, 5, 0, 2, respectively, and u_{l-3}, \dots, u_3 using pattern 602. If $f(v_3) = 5$, then label $u_2, u_1, u_l, u_{l-1}, u_{l-2}$ by 0, 3, 5, 2, 4, respectively, and u_{l-3}, \dots, u_3 using pattern 024.

Next we assume $f(v_2) \in \{1, 3, 5\}$ and $|\{f(v_1), f(v_3)\} \cap \{1, 3, 5\}| \leq 1$. We assign u_2 a label from $\{1, 3, 5\} \setminus \{f(v_1), f(v_2), f(v_3)\}$ and then assign u_1 a label from $\{1, 3, 5\} \setminus \{f(u_2), f(v_2)\}$. Let $x \in \{1, 3, 5\}$. Denote by L_x the set of available neighbor labels for x in $\{0, 2, 4, 6\}$. Note that for each of $f(u_1)$ and $f(u_2)$, there are two available neighbor labels in $\{0, 2, 4, 6\}$. By Lemma 2.4, $|L_{f(u_1)} \setminus L_{f(u_2)}| = 2$ or 1. In the former case, we can choose an available neighbor label $a \in \{0, 2, 4, 6\}$ for $f(u_1)$ and an available neighbor label $b \in \{0, 2, 4, 6\}$ for $f(u_2)$ such that $a \neq b$. Choose $c \in \{0, 2, 4, 6\} \setminus \{a, b, d\}$, where $L_{f(u_1)} = \{a, d\}$. In the latter case, let $b \in L_{f(u_1)} \setminus L_{f(u_2)}$, $a \in L_{f(u_1)} \setminus \{b\}$ and $c \in \{0, 2, 4, 6\} \setminus \{a, b, d\}$, where $L_{f(u_1)} = \{b, d\}$. In both cases we label u_3, u_4, \dots, u_l using pattern bca .

Finally, we assume $f(v_2) \in \{0, 2, 4, 6\}$. We assign u_2 a label $a \in \{0, 2, 4, 6\} \setminus \{f(v_1), f(v_2), f(v_3)\}$. Then we assign u_1, u_l labels d_1, d_2 from $\{1, 3, 5\}$, respectively, such that a is not an available neighbor label of d_2 . Let b be an available neighbor labels in $\{0, 2, 4, 6\}$ for d_2 . Choose $c \in \{0, 2, 4, 6\} \setminus \{a, b, f(v_2)\}$. We label u_3, u_4 by c, b , respectively, and u_5, \dots, u_{l-1} using pattern acb .

In each case above, by Lemmas 2.6 and 2.7, we obtain a 6- $L(2, 1)$ -labeling f_1 of K with the desired property. ■

6 Proof of Theorem 1.4

Throughout this section G is an outer plane graph with $\Delta = 3$. A path $P = v_1 v_2 \dots v_t$ of G is called a *branch* if $d(v_1) \geq 3$, $d(v_t) \geq 3$ and $d(v_i) = 2$ for $2 \leq i \leq t-1$. For two blocks A and B of G , define $d(A, B) = \min\{d(x, y) : x \in V(A), y \in V(B)\}$, where $d(x, y)$ is the distance in G between x and y . Let B_1 and B_2 be two blocks of G such that $d(B_1, B_2)$ is minimized. Since $\Delta = 3$, B_1 is joined to B_2 by a branch of length at least one.

Proof of Theorem 1.4. Suppose to the contrary that not every outerplanar graph with maximum degree 3 satisfies $\lambda \leq 6$. Let G be a smallest counterexample. That is, G is an outerplanar graph of maximum degree 3 having no 6- $L(2, 1)$ -labelings such that $|V(G)|$ is minimum. Clearly, $|V(G)| \geq 4$ and G is connected by the minimality of G . We prove the following claim first.

Claim. G is 2-connected.

Proof of the Claim. Suppose G is not 2-connected. Since G is connected, it has a cut edge. By Lemma 2.2, G has no vertex of degree 1. Thus G consists of blocks and branches connecting blocks. We construct a graph X as follows: $V(X)$ is the set of blocks of G ; for $x, y \in V(X)$, let B_x and B_y denote the blocks of G corresponding to x and y , respectively. Vertex x is adjacent to vertex y in X if and only if block B_x is connected to block B_y by a branch of G . It is obvious that X is a tree. Let u be a vertex of X with degree one and v the unique neighbor of u in X . Let $P = v_1 v_2 \dots v_k$ be the branch connecting B_u and B_v , where $v_1 \in V(B_v)$ and $v_k \in V(B_u)$. Then $k \geq 2$ as $\Delta = 3$. Let G_1 denote the graph obtained from $G - V(B_u)$ by deleting v_2, \dots, v_{k-1} . Since $|V(B_u)| \geq 3$, we have $|V(G_1)| \leq |V(G)| - 2$. By the choice of G , G_1 has a 6- $L(2, 1)$ -labeling. By Lemma 2.2, $G - V(B_u)$ has a 6- $L(2, 1)$ -labeling.

If $k = 2$, let u_1 and u_2 be two neighbors of v_1 in B_v and let B'_u be the graph induced by $V(G_u) \cup \{v_1, u_1, u_2\}$. Note that $u_1 v_1 u_2$ has a 6- $L(2, 1)$ -labeling which, by Theorems 5.4 and 4.6, can be extended to a 6- $L(2, 1)$ -labeling of B'_u . Thus G has a 6- $L(2, 1)$ -labeling, contradicting our assumption.

Thus we assume $k \geq 3$. Define $G_2 = G - (V(G_1) \cup \{v_2, \dots, v_{k-3}\})$. That is, G_2 is obtained from the block B_u by adding the path $v_k v_{k-1} v_{k-2}$. Note that v_{k-1} and v_{k-2} have been assigned labels from $[0, 6]$. To prove Theorem 1.4, it is sufficient to prove that the existing 6- $L(2, 1)$ -labeling of $v_{k-2} v_{k-1}$ can be extended to a 6- $L(2, 1)$ -labeling of G_2 . To apply Theorem 5.4, we construct a graph G_2^* obtained from G_2 by adding to G_2 a new vertex w together with an edge joining w and v_{k-1} . Now we label w as follows: if $f(v_{k-1}) \in \{0, 2, 4, 6\}$, then w is assigned a label from $\{0, 2, 4, 6\} \setminus \{f(v_{k-2}), f(v_{k-1})\}$; if $f(v_{k-1}) \in \{1, 3, 5\}$, then w is assigned a label from $\{1, 3, 5\} \setminus \{f(v_{k-2}), f(v_{k-1})\}$. Clearly, our labeling of $w v_{k-1} v_{k-2}$ is a 6- $L(2, 1)$ -labeling. By Theorems 5.4 and 4.6, the 6- $L(2, 1)$ -labeling of $w v_{k-1} v_{k-2}$ can be extended to a 6- $L(2, 1)$ -labeling f of G_2^* . Clearly, $f|_{G_2}$ is a 6- $L(2, 1)$ -labeling of G_2 , which together with a 6- $L(2, 1)$ -labeling of $G - V(B_u)$ gives a 6- $L(2, 1)$ -labeling of G , a contraction. Therefore, G is 2-edge-connected. Since G is an outer plane graph with $\Delta(G) = 3$, G must be 2-connected. \square

By the Claim, G is 2-connected. If $|V(G)| = 4$, then G is isomorphic to the complete K_4 with one edge removed, and so G has a 6- $L(2, 1)$ -labeling. Assume $|V(G)| \geq 5$. Since $\Delta = 3$, G contains at least two vertices of degree 3. It follows that G contains two adjacent faces F_1 and F_2 . If both F_1 and F_2 are two 3-faces, then by Lemma 2.1, G is isomorphic to the complete K_4 with one edge removed, contracting $|V(G)| \geq 5$. Thus G contains a face F with $|\partial F| \geq 4$. Denote $\partial F = v_1 v_2 \dots v_l$. We assign labels from $[0, 6]$ to the vertices of ∂F in the follow way.

If $l \equiv 0 \pmod{3}$, then label v_1, v_2, \dots, v_l using pattern 0 2 4; if $l \equiv 1 \pmod{3}$, then label v_1 by 3 and v_2, \dots, v_l using pattern 6 4 0; if $l \equiv 2 \pmod{3}$, then label v_1, v_2 by 3, 1, respectively, and v_3, \dots, v_l using pattern 4 2 0. By Lemmas 2.6 and 2.7, the labeling of ∂F is a cycle-extendable 6- $L(2, 1)$ -labeling of type 1. If $G = F$, we are done. Assume $G \neq F$. Let F' be a face adjacent to F . Since G is an outer plane graph and G is 2-connected, $|E(F) \cap E(F')| = 1$. We assume, without loss of generality, that $E(F) \cap E(F') = v_2 v_3$. By Theorem 4.6, the 6- $L(2, 1)$ -labeling of $v_1 v_2 v_3 v_4$ can be extended to a path-extendable 6- $L(2, 1)$ -labeling of $\partial F' - v_2 v_3$. Repeating the extendable procedure above, finally we obtain that G has a 6- $L(2, 1)$ -labeling. This contradiction proves the upper bound in Theorem 1.4.

In particular, we have $\lambda(G(l)) \leq 6$. This and Theorem 3.6 imply $\lambda(G(l)) = 6$ when $l \geq 4$ is not a multiple of 3. \blacksquare

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